

Fluctuations and skewness of the current in the partially asymmetric exclusion process

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2008 J. Phys. A: Math. Theor. 41 365003

(<http://iopscience.iop.org/1751-8121/41/36/365003>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.150

The article was downloaded on 03/06/2010 at 07:09

Please note that [terms and conditions apply](#).

Fluctuations and skewness of the current in the partially asymmetric exclusion process

Sylvain Prolhac

Institut de Physique Théorique, CEA, IPhT, F-91191 Gif-sur-Yvette, France
and
CNRS, URA 2306, F-91191 Gif-sur-Yvette, France

Received 28 May 2008, in final form 11 July 2008

Published 30 July 2008

Online at stacks.iop.org/JPhysA/41/365003

Abstract

We use functional Bethe ansatz equations to calculate the cumulants of the total current in the partially asymmetric exclusion process. We recover known formulae for the first two cumulants (mean value of the current and diffusion constant) and obtain an explicit finite-size formula for the third cumulant. The expression for the third cumulant takes a simple integral form in the limit where the asymmetry scales as the inverse of the square root of the size of the system, which corresponds to a natural separation between weak and strong asymmetry.

PACS numbers: 05.40.-a, 05.60.-k

1. Introduction

The asymmetric simple exclusion process (ASEP) is one of the most simple examples of a stochastic interacting particles model with an out of equilibrium stationary state. It features classical particles hopping on a lattice and interacting through hard core exclusion. It can be seen as a growth model with deposition and evaporation of particles and is thus a discrete version of a system evolving by the Kardar–Parisi–Zhang (KPZ) equation. The one-dimensional model has been much studied in the past [1–5]. It is known to be exactly solvable through various methods including the Bethe ansatz [6, 7], which uses the underlying integrability of the Markov matrix of the model (similar to the Hamiltonian of an XXZ spin chain with twisted boundary conditions), and the matrix product ansatz method [8, 9], which consists in writing the stationary state eigenvector as a trace of product of matrices.

The fluctuations of the current have received much attention [10–12]. In the long time limit, the system reaches a stationary state, independent of the initial configuration for finite-size systems. A manifestation of the fact that the system is not at equilibrium in the stationary state is the existence of a current of particles flowing through the system. In this stationary state, the mean value J of the current is simply proportional to the asymmetry of the rates at which the particles hop to the right and to the left, which can be thought as a driving field.

The fluctuations of the current describe how the current differs from its mean value, that is the probability of finding a current j different from J . But finding the whole probability density function can be difficult. Instead, one can study its first cumulants. The second cumulant, related to the diffusion constant, describes the width of the probability density whereas the third cumulant represents the asymmetry and the non-Gaussianity of the probability density. The fourth cumulant describes how much the peak of the distribution is sharp.

The fluctuations of the current have been studied for different variants of the ASEP: finite-size open lattice [12], systems with second-class particles [11, 13], infinite lattice [14, 15], etc. For the totally asymmetric exclusion process, in which the particles only hop in one direction (TASEP), on a finite-size lattice with periodic boundary conditions, all the cumulants of the probability distribution of the current have been calculated by Bethe ansatz, using a simplification of the Bethe ansatz equations that occur only in this case. For the partially asymmetric exclusion process on a ring, a particular scaling limit of the probability distribution function of the current was calculated using a thermodynamic limit of the Bethe ansatz equations [16]. However, only the first two cumulants were known for finite-size systems [17, 18]. In the present paper, we introduce a method that allows us to calculate exact expressions for the first cumulants. We use it to get the diffusion constant with very few calculations, and obtain an explicit formula (58) for the third cumulant. We also derive a rather simple integral formula (69) for the third cumulant in the scaling limit where the asymmetry goes to zero as the inverse of the square root of the size of the system. This formula is similar to the one obtained in [17] for the diffusion constant in the same scaling limit.

The structure of the paper is as follows. In section 2, we recall the functional Bethe ansatz equations used in [18] to calculate the two first cumulants of the total current. In section 3, we show how to solve perturbatively these equations, giving a way to calculate explicitly the cumulants one after the other. We obtain easily the two first cumulants (mean value of the current and diffusion constant). For the diffusion constant, our derivation is much simpler than in [17, 18]. In section 5, after a more involved calculation, we obtain an explicit expression for the third cumulant. We show that for the cases of either the symmetric exclusion process (SSEP, the particle hop on both sides with the same rate) or TASEP it reduces to known formulae. We finally write the third cumulant formula in an integral form in the scaling limit where the asymmetry goes to zero as the inverse of the square root of the size of the system.

2. Bethe ansatz for the fluctuations of the current

We consider in this paper the partially asymmetric simple exclusion process (PASEP) on a one-dimensional lattice with periodic boundary conditions. Each of the L sites can be empty or occupied by at most one of the n particles. The configuration space of the model has then dimension $\Omega = \binom{L}{n}$. On this configuration space, we define the following Markov process: each particle can hop to the right with rate p (that is, with probability $p dt$ for an infinitesimal time interval dt) and to the left with rate q provided that the target site is empty. Otherwise, the particle cannot move. This dynamics can be encoded in the Markov matrix $M = M_0 + pM_1 + qM_{-1}$, whose diagonal part M_0 contains the exit rates from each configuration, and M_1 (resp. M_{-1}) the incoming rates obtained by moving one particle to the right (resp. to the left).

Let Y_t be the total distance covered by all the particles between time 0 and t . Y_t is thus the integrated current between time 0 and t . It has been shown [10, 18, 19] that the generating function of the cumulants of Y_t in the long time limit can be obtained from the diagonalization of a deformation of the Markov matrix of the system: $M(\gamma) = M_0 + p e^\gamma M_1 + q e^{-\gamma} M_{-1}$,

introducing the parameter γ which can be seen as a fugacity associated with the leaps of the particles. More precisely, the eigenvalue $E(\gamma)$ of $M(\gamma)$ corresponding to the stationary state of the system, that is its eigenvalue with maximal real part, is given by

$$E(\gamma) = \lim_{t \rightarrow \infty} \frac{\log \langle e^{\gamma Y_t} \rangle}{t} = J\gamma + \frac{\Delta}{2}\gamma^2 + \frac{E_3}{6}\gamma^3 + \dots, \quad (1)$$

where J is the total current and Δ is the diffusion constant.

$$J = \lim_{t \rightarrow \infty} \frac{\langle Y_t \rangle}{t} \quad (2)$$

$$\Delta = \lim_{t \rightarrow \infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} \quad (3)$$

$$E_3 = \lim_{t \rightarrow \infty} \frac{\langle Y_t^3 \rangle - 3\langle Y_t \rangle \langle Y_t^2 \rangle + 2\langle Y_t \rangle^3}{t}. \quad (4)$$

The diagonalization of $M(\gamma)$ can be performed using the Bethe ansatz, because of the underlying integrability of the model. We are now going to recall the functional equations derived in [18] for the calculation of $E(\gamma)$. Defining the asymmetry parameter x by

$$x = \frac{q}{p}, \quad (5)$$

the Bethe equations of the system are

$$e^{L\gamma} \left(\frac{1 - y_i}{1 - xy_i} \right)^L + \prod_{j=1}^n \frac{y_i - xy_j}{xy_i - y_j} = 0. \quad (6)$$

The Bethe roots y_i $1 \leq i \leq n$ depend on both x and γ . They can be used to write the corresponding eigenvector as a sum over the $n!$ permutations of the particles. Introducing the polynomial

$$Q(t) = \prod_{i=1}^n (t - y_i), \quad (7)$$

whose zeros are the Bethe roots, it can be shown [18] that the Bethe equations (6) can be rewritten as functional equations

$$Q(t)R(t) = e^{L\gamma} (1 - t)^L Q(xt) + (1 - xt)^L x^n Q(t/x), \quad (8)$$

where Q is a polynomial of degree n (with a coefficient of highest degree equal to 1) and R is a polynomial of degree L . Both Q and R must be determined by solving equation (8). This equation is known as Baxter's scalar TQ equation [20–23]. Equation (8) sets the value of $R(0)$,

$$R(0) = x^n + e^{L\gamma}, \quad (9)$$

and the behavior of $R(t)$ when $t \rightarrow \infty$,

$$R(t) \sim (x^L + x^n e^{L\gamma}) (-1)^L t^L \quad \text{when } t \rightarrow \infty. \quad (10)$$

When $\gamma = 0$, the solution of the Bethe equations (6) corresponding to the stationary state is $y_i = 0$ for all i . Equivalently, the corresponding solution of (8) is $Q(t) = t^n$ and $R(t) = x^n(1 - t)^L + (1 - xt)^L$. Expanding $Q(t)$ and $R(t)$ near $\gamma = 0$, we get

$$Q(t) = t^n + \mathcal{O}(\gamma) \quad (11)$$

and

$$R(t) = x^n(1 - t)^L + (1 - xt)^L + \mathcal{O}(\gamma). \quad (12)$$

The stationary state has another interesting property: it is a state with zero momentum, which leads to [18]

$$x^n Q(1/x) = e^{n\gamma} Q(1) \tag{13}$$

or, in terms of the polynomial R ,

$$R(1) = e^{n\gamma} (1-x)^L. \tag{14}$$

The latter equation does not provide additional information beside (12), but it is useful to make some calculations easier.

Finally, with Q and R solution of (8), the eigenvalue of $M(\gamma)$ corresponding to the stationary state is given by [18]

$$\frac{E(\gamma)}{p} = (1-x) \left(\frac{Q'(1)}{Q(1)} - \frac{1}{x} \frac{Q'(1/x)}{Q(1/x)} \right) = -Lx - (1-x) \frac{R'(1)}{R(1)}. \tag{15}$$

Equation (8) could be solved perturbatively near $\gamma = 0$ right now. Inserting the solution into (15), it would give us the first cumulants of the current. This was done in [18] up to order 2. Here, we will rewrite the previous equations in a different way before making the perturbative expansion near $\gamma = 0$. It will make the calculations much simpler than in [18], and allow us to calculate the third cumulant. We divide both sides of the functional equation (8) by $(1-t)^L(1-xt)^L Q(t)$,

$$\frac{R(t)}{(1-t)^L(1-xt)^L} = \frac{A(t)}{(1-t)^L} + \frac{1}{(1-xt)^L} \frac{x^n e^{L\gamma}}{A(xt)}, \tag{16}$$

where we defined

$$A(t) = x^n \frac{Q(t/x)}{Q(t)}. \tag{17}$$

From this definition of $A(t)$, and using (13) and the fact that Q is of degree n , we know the value of $A(t)$ in $t = 0$, $t = 1$ and in the limit $t \rightarrow \infty$,

$$A(0) = x^n, \quad A(1) = e^{n\gamma} \quad \text{and} \quad \lim_{t \rightarrow \infty} A(t) = 1. \tag{18}$$

From (11), we also know that

$$A(t) = 1 + \mathcal{O}(\gamma). \tag{19}$$

In terms of $A(t)$, expression (15) for the eigenvalue rewrites, using the value of $A(1)$ (18),

$$\frac{E(\gamma)}{p} = -(1-x) e^{-n\gamma} A'(1). \tag{20}$$

3. Perturbative solution of the functional equation

In this section, we will first show how to eliminate $R(t)$ from equation (16), leaving us with a closed equation for $A(t)$. We will see how we can solve this equation perturbatively near $\gamma = 0$. Then, we will reformulate this solution to make the explicit calculation of the cumulants easier.

3.1. Perturbative solution for $A(t)$

From (19), we can write the expansion of $A(t)$ near $\gamma = 0$ as

$$A(t) = 1 + \sum_{k=1}^{\infty} A_k(t)\gamma^k. \tag{21}$$

But $Q(t)$ is normalized such that $Q(t) - t^n$ is a polynomial of degree $n - 1$. Thus, from (17), the $A_k(t)$ are polynomials in $1/t$ of degree kn without a constant term ($A_k(t) \rightarrow 0$ when $t \rightarrow \infty$).

From now on, we will look at a perturbative solution of (16) for small γ . $R(t)$, which is a polynomial in t , can be seen as a formal series in t and γ with only non-negative powers in t . The lhs of equation (16) is then also a formal series in t and γ with only non-negative powers in t . But we also know that, at each (nonzero) order in γ , $A(t)$ has only negative powers in t so that the rhs of equation (16) has both negative and non-negative powers in t . Thus, equation (16) simply means that the negative powers in t in equation (16) cancel. We will write this as

$$\frac{A(t)}{(1-t)^L} + \frac{1}{(1-xt)^L} \frac{x^n e^{L\gamma}}{A(xt)} = \mathcal{O}(t^0), \tag{22}$$

which will mean: at each order in γ , the lhs of (22) is finite when $t \rightarrow 0$. We have eliminated $R(t)$ from equation (16). We will see that equation (22) still determines $A(t)$ uniquely.

From now on, every expansion in powers of t will have to be understood as an expansion of a formal series in powers of γ followed, at each order in γ , by an expansion in powers of t . In the following, we will use the notation $[f(t)]_{(k)} \equiv [f]_{(k)}$ to refer to the coefficient of the term t^k in the expansion of the formal series $f(t)$. We also introduce the notation $[f(t)]_{(-)}$ which will represent the negative powers of f in t (that is terms t^k with $k < 0$). In contrast, $[f(t)]_{(+)}$ will represent the non-negative powers of f in t (terms t^k with $k \geq 0$).

To solve equation (22), we will write it in the slightly more complicated form

$$\Delta_x \left(\frac{A(t)}{(1-t)^L} \right) = - \frac{x^n}{(1-xt)^L} \left(A(xt) + \frac{e^{L\gamma}}{A(xt)} \right) + \mathcal{O}(t^0), \tag{23}$$

with Δ_x being the operator which acts on an arbitrary function $u(t) = \sum_{k \in \mathbb{Z}} [u]_{(k)} t^k$ as

$$(\Delta_x u)(t) = u(t) - x^n u(xt) = \sum_{k \in \mathbb{Z}} (1 - x^{n+k}) [u]_{(k)} t^k. \tag{24}$$

As Δ_x gives 0 when applied to t^{-n} , the equation

$$(\Delta_x u)(t) = v(t) \tag{25}$$

can be formally solved as

$$u(t) = (\Delta_x^{-1} v)(t) = \sum_{k \neq -n} \frac{[v]_{(k)} t^k}{1 - x^{n+k}} - \frac{b}{t^n} \tag{26}$$

for any formal series v which has no term t^{-n} . b is an arbitrary coefficient which is not constrained by equation (25). We can use this to invert Δ_x in (23). Recalling that $A(t) \rightarrow 1$ when $t \rightarrow \infty$ and that $A(t) - 1$ has only negative powers in t , we obtain

$$A(t) = 1 - [(1-t)^L \tilde{g}(t)]_{(-)} - b \left[\frac{(1-t)^L}{t^n} \right]_{(-)}, \tag{27}$$

with

$$\tilde{g}(t) = \left[\Delta_x^{-1} \left(\frac{x^n}{(1-xt)^L} \left(A(xt) + \frac{e^{L\gamma}}{A(xt)} \right) \right) \right]_{(-)}. \tag{28}$$

The term containing b is not constrained by equation (23). The condition necessary to invert Δ_x in equation (23), which is that the rhs of (23) must not contain a term t^{-n} , implies that $\tilde{g}(t)$ does not contain a term t^{-n} .

Written like that, equation (27) contains A on both sides (through \tilde{g} on the rhs). However, equation (27) gives a recursive solution for $A(t)$ order by order in powers of γ . Indeed, if we take the term of order γ^k in this equation, the lhs depends only on $A_k(t)$ whereas the rhs depends only on the $A_j(t)$ with $j < k$: the $A_k(t)$ cancels out at order k in the expansion in powers of γ of

$$A(xt) + \frac{e^{L\gamma}}{A(xt)} = 1 + \sum_{k=1}^{\infty} A_k(xt)\gamma^k + \frac{1 + (e^{L\gamma} - 1)}{1 + \sum_{k=1}^{\infty} A_k(xt)\gamma^k}. \quad (29)$$

Thus, equation (27) allows us to recursively compute all the orders in γ starting from (19). The parameter b in (27) can be set from the value (18) of $A(1)$.

Using equations (27), (19) and (18), we can do the calculation for the first cumulants. But we will first turn these equations into another form which will make the calculations a little simpler.

We must emphasize that our method involves only algebraic manipulations of formal series in two parameters, namely γ and t . The Bethe roots $y_i(\gamma)$ (which are the zeros of the polynomial $Q(t)$ and the poles of the rational function $A(t)$) have completely disappeared since we did the γ expansion first: the $A_k(t)$ have only a pole of order kn in $t = 0$. We do not have to follow the $y_i(\gamma)$ as a function of γ : we only use some algebraic properties of $A(t)$.

3.2. A simpler formulation for the perturbative solution

We will now eliminate $A(t)$ from our recursive equations, and work on the auxiliary quantity $\tilde{g}(t)$. Noting that $A^{-1}(xt) - 1$ has also only negative powers in t , we find from equations (22) and (27) that $x^n e^{L\gamma} A^{-1}(xt)$ can be written in terms of $\tilde{g}(t)$ as

$$x^n e^{L\gamma} A^{-1}(xt) = x^n e^{L\gamma} + [(1 - xt)^L \tilde{g}(t)]_{(-)} + b \left[\frac{(1 - xt)^L}{t^n} \right]_{(-)}. \quad (30)$$

We can absorb the b terms of $A(t)$ and $x^n e^{L\gamma} A^{-1}(xt)$ into \tilde{g} , defining

$$g(t) = \tilde{g}(t) + \frac{b}{t^n}. \quad (31)$$

b is thus the coefficient of t^{-n} in $g(t)$ as $\tilde{g}(t)$ has no term t^{-n} . In the following, we will both need using $g(t)$ and $\tilde{g}(t)$. The expressions for $A(t)$ and $x^n e^{L\gamma} A^{-1}(xt)$ become

$$A(t) = 1 - [(1 - t)^L g(t)]_{(-)} \quad (32)$$

$$x^n e^{L\gamma} A^{-1}(xt) = x^n e^{L\gamma} + [(1 - xt)^L g(t)]_{(-)}. \quad (33)$$

From (28), at order k in γ $g(t)$ is a polynomial in $1/t$ of degree kn without a constant term, as are the $A_k(t)$. We also see that $g(t)$ is equal to 0 when $\gamma = 0$ (as $A(t) = 1$ when $\gamma = 0$), giving the expansion

$$g(t) = \sum_{k=1}^{\infty} g_k(t)\gamma^k. \quad (34)$$

To find a closed equation for $g(t)$, we eliminate $A(t)$ between (32) and (33), using $(A(t))(x^n e^{L\gamma} A^{-1}(t)) = x^n e^{L\gamma}$,

$$[(1 - t)^L g(t/x)]_{(-)} - x^n e^{L\gamma} [(1 - t)^L g(t)]_{(-)} - [(1 - t)^L g(t)]_{(-)} [(1 - t)^L g(t/x)]_{(-)} = 0, \quad (35)$$

which can be rewritten as

$$g(t/x) - x^n e^{L\gamma} g(t) = \frac{[(1-t)^L g(t)]_{(-)} [(1-t)^L g(t/x)]_{(-)}}{(1-t)^L} + \mathcal{O}(t^0). \tag{36}$$

As $g(t)$ is of order γ , this allows us to solve order by order in powers of γ . We can simplify the rhs of the previous equation, writing

$$\begin{aligned} & \frac{[(1-t)^L g(t)]_{(-)} [(1-t)^L g(t/x)]_{(-)}}{(1-t)^L} \\ &= \frac{((1-t)^L g(t) - [(1-t)^L g(t)]_{(+)}) ((1-t)^L g(t/x) - [(1-t)^L g(t/x)]_{(+)})}{(1-t)^L} \\ &= (1-t)^L g(t)g(t/x) - g(t/x)[(1-t)^L g(t)]_{(+)} - g(t)[(1-t)^L g(t/x)]_{(+)} + \mathcal{O}(t^0) \\ &= -(1-t)^L g(t)g(t/x) + g(t/x)[(1-t)^L g(t)]_{(-)} + g(t)[(1-t)^L g(t/x)]_{(-)} + \mathcal{O}(t^0) \end{aligned} \tag{37}$$

and we obtain

$$g(t/x) - x^n e^{L\gamma} g(t) = -(1-t)^L g(t)g(t/x) + g(t)[(1-t)^L g(t/x)]_{(-)} + g(t/x)[(1-t)^L g(t)]_{(-)} + \mathcal{O}(t^0) \tag{38}$$

or, at order r in γ , multiplying t by x

$$\begin{aligned} (\Delta_x g_r)(t) &= x^n \sum_{k=1}^{r-1} \frac{L^k}{k!} g_{r-k}(xt) - \sum_{k=1}^{r-1} ((1-xt)^L g_{r-k}(xt)g_k(t) \\ &\quad - g_k(t)[(1-xt)^L g_{r-k}(xt)]_{(-)} - g_{r-k}(xt)[(1-xt)^L g_k(t)]_{(-)} + \mathcal{O}(t^0), \end{aligned} \tag{39}$$

where we used once again the operator Δ_x (24). Using the formal inversion formula (26) for Δ_x , we see that the previous equation for $g(t)$ can be solved if

$$\begin{aligned} & \sum_{k=1}^{r-1} \left[x^n \frac{L^k}{k!} g_{r-k}(xt) - (1-xt)^L g_{r-k}(xt)g_k(t) + g_k(t) \right. \\ & \quad \left. \times [(1-xt)^L g_{r-k}(xt)]_{(-)} + g_{r-k}(xt)[(1-xt)^L g_k(t)]_{(-)} \right]_{(-n)} = 0, \end{aligned} \tag{40}$$

which is a consequence of the fact that equation (39) holds: it must be true if (8) has a solution verifying (11) and (12). We obtain

$$\begin{aligned} \tilde{g}_r(t) &= \sum_{k=1}^{r-1} \sum_{\substack{l=1 \\ (l \neq n)}}^{(r-k)n} \frac{x^n t^{-l}}{1-x^{n-l}} \left[\frac{L^k}{k!} g_{r-k}(xt) \right]_{(-l)} - \sum_{k=1}^{r-1} \sum_{\substack{l=1 \\ (l \neq n)}}^{rn} \\ & \times \frac{[(1-xt)^L g_{r-k}(xt)g_k(t) - g_k(t)[(1-xt)^L g_{r-k}(xt)]_{(-)} - g_{r-k}(xt)[(1-xt)^L g_k(t)]_{(-)}]_{(-l)}}{(1-x^{n-l})t^l} \end{aligned} \tag{41}$$

and

$$g_r(t) = \frac{b_r}{t^n} + \tilde{g}_r(t). \tag{42}$$

The coefficient b_r is the term of order γ^r in b . Again, the value (18) of $A(1)$ sets b_r . We note that this formula does not require to divide by a formal series in γ , unlike (27) where we had to divide by $A(xt)$. The nonlinearity of the recurrence formula is then simpler here; it consists only in a product of two series in γ .

4. Explicit calculations for the two first cumulants of the current

In this section, after expressing the generating function of the cumulants of the current in terms of the perturbative solution described in the previous section, we will calculate explicitly the two first cumulants.

4.1. Expression of the stationary state eigenvalue

We will now express the stationary state eigenvalue using the function $g(t)$. As $g(t)$ has only negative powers in t , expression (32) for $A(t)$ in terms of $g(t)$ can be written as

$$A(t) = 1 - \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} [g]_{(-i)} \binom{L}{j} (-1)^j t^{j-i}. \quad (43)$$

Using the binomial coefficients formulae (A.1) and (A.2), we get for the eigenvalue

$$\frac{E(\gamma)}{p} = \frac{1-x}{e^{n\gamma}} \sum_{j=1}^L \binom{L}{j} (-1)^j \frac{j(L-j)}{L(L-1)} [g]_{(-j)}. \quad (44)$$

This expression for $E(\gamma)$ is of order γ , as is $g(t)$, which means $E(\gamma = 0) = 0$, as expected for the stationary state.

The last equation allows us to calculate the eigenvalue at any order in γ if we know $g(t)$ at the corresponding order. $g(t)$ is obtained by the recurrence equation (41) in which b_r is set using the known value (18) of $A(1)$. Thus, in the calculation of the eigenvalue b_r can be eliminated from (44) where it appears through $g(t) = \tilde{g}(t) + \frac{b}{p}$. If we want to calculate the eigenvalue at order 3, we will not need b_3 (but we will still need b_1 and b_2 as the recurrence equation (41) involves $g(t)$ and not only $\tilde{g}(t)$). Using the value (18) of $A(1)$,

$$\sum_{j=1}^L \binom{L}{j} (-1)^j \frac{j}{L} [g]_{(-j)} = e^{n\gamma} - 1, \quad (45)$$

which gives for $b = \sum_{r=1}^{\infty} b_r \gamma^r$,

$$\frac{b}{v} = \frac{e^{n\gamma} - 1}{n} - \sum_{j=1}^L \binom{L}{j} (-1)^j \frac{j}{nL} [\tilde{g}]_{(-j)}, \quad (46)$$

with

$$v \equiv (-1)^n \frac{L}{\binom{L}{n}}. \quad (47)$$

We finally get for the eigenvalue, putting (44) and (46) together,

$$\frac{E(\gamma)}{p} = \frac{1-x}{e^{n\gamma}} \left(\sum_{j=1}^L \binom{L}{j} (-1)^j \frac{j(n-j)}{L(L-1)} [\tilde{g}]_{(-j)} + \frac{L-n}{L-1} (e^{n\gamma} - 1) \right). \quad (48)$$

We will now use (41), (46) and (48) to calculate $E(\gamma)$ up to order 2 in γ , recovering known results for the mean value of the current and for the diffusion constant.

4.2. Calculation of the mean value of the current

From (1), the mean value J of the current is given by

$$\frac{J}{p} = \left(\frac{dE(\gamma)}{d\gamma} \right)_{|\gamma=0}. \quad (49)$$

We only need $E(\gamma)$ at the first order in γ . In this case, the two complicated terms with the sums in (41) do not contribute and $\tilde{g}_1(t) = 0$. Using (48), the eigenvalue reads

$$\frac{J}{p} = (1-x) \frac{n(L-n)}{L-1}. \quad (50)$$

We did not need b_1 to calculate J , but we will need it for the next orders. Using (46) we find $b_1 = v$ and

$$g_1(t) = \frac{v}{t^n}. \quad (51)$$

4.3. Calculation of the diffusion constant

From (1), the diffusion constant Δ is given by

$$\frac{\Delta}{p} = \left(\frac{d^2 E(\gamma)}{d\gamma^2} \right)_{|\gamma=0}. \quad (52)$$

The recurrence (41) gives

$$\tilde{g}_2(t) = v^2 \sum_{k=1}^{n-1} \frac{\binom{L}{2n-k} (-t)^{-k}}{1-x^{k-n}} - v^2 \sum_{k=n+1}^{2n} \frac{\binom{L}{2n-k} (-t)^{-k}}{1-x^{k-n}}. \quad (53)$$

The eigenvalue is, using (48) and (A.4),

$$\frac{\Delta}{p} = 2(1-x) \frac{L}{L-1} \sum_{k>0} k^2 \frac{\binom{L}{n+k} \binom{L}{n-k}}{\binom{L}{n}^2} \frac{1+x^k}{1-x^k}. \quad (54)$$

The derivation of this expression is much simpler than the previous derivation using the Bethe ansatz [18]. It uses a little more formalism, but nearly no calculation is needed, contrary to the previous derivation in which many unexpected simplifications occurred in the end of the calculation.

Condition (40) necessary for the consistency of equation (39) is easily checked

$$Lx^n [g_1(t)]_{(-n)} - [(1-t)^L g_1(t) g_1(t/x)]_{(-n)} = 0. \quad (55)$$

As for the first order, we did not need b_2 to calculate the diffusion constant, but we will need it for the third order. We set b_2 using equation (46) and the binomial formula (A.4),

$$\frac{b_2}{v} = \frac{L}{2} + \frac{v^2}{L} \sum_{k=1}^n \binom{L}{n+k} \binom{L}{n-k} \frac{1+x^k}{1-x^k} \quad (56)$$

and obtain for $g_2(t)$,

$$g_2(t) = v^2 \sum_{k=1}^{n-1} \frac{\binom{L}{2n-k} (-t)^{-k}}{1-x^{k-n}} - v^2 \sum_{k=n+1}^{2n} \frac{\binom{L}{2n-k} (-t)^{-k}}{1-x^{k-n}} + \frac{Lv}{2t^n} + \frac{v^3}{Lt^n} \sum_{k=1}^n \binom{L}{n+k} \binom{L}{n-k} \frac{1+x^k}{1-x^k}. \quad (57)$$

5. Third cumulant of the current

This section is devoted to the third cumulant of the current in the stationary state of PASEP. First, we will explain the steps needed to derive expression (58) for the third cumulant, leaving the complete proof to appendix B. We will then show that we recover known formulae in the case of TASEP and SSEP. Finally, we will write the expression for the third cumulant as a double integral in the scaling limit where the asymmetry goes to zero as the inverse of the square root of the size of the system.

5.1. Exact formula

We will now calculate the third cumulant of the current. Without loss of generality, we can suppose that $L \leq 2n$ because of the particle-hole symmetry. It will be easier because all the sums will have their bounds between 0 and $2n$. The steps are the same as for the two first orders: first, calculate $\tilde{g}_3(t)$ using the recurrence relation (41). Then, insert it into (48) to get the eigenvalue. This time, as we do not want the next order, we will not need b_3 . The only big difference will be that these two steps use now longer expressions involving double sums (instead of simple sums for the second order and no sum at all for the first order). In the end of the calculation, which is detailed in Appendix B, we find that the third cumulant E_3 is given by

$$\begin{aligned} \frac{(L-1)E_3}{p(1-x)L^2} &= 6 \sum_{i>0} \sum_{j>0} \frac{\binom{L}{n+i} \binom{L}{n-i} \binom{L}{n+j} \binom{L}{n-j}}{\binom{L}{n}^4} (i^2 + j^2) \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} \\ &\quad - 3 \sum_{i>0} \sum_{j>0} \frac{\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j}}{\binom{L}{n}^3} \\ &\quad \times (i^2 + ij + j^2) \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} - 3 \sum_{i>0} \frac{\binom{L}{n+i} \binom{L}{n-i}}{\binom{L}{n}^2} (i^2) \left(\frac{1+x^i}{1-x^i} \right)^2 \\ &\quad + \frac{3n(L-n)}{2(2L-1)} \frac{\binom{2L}{2n}}{\binom{L}{n}^2} - \frac{n(L-n)}{(3L-1)} \frac{\binom{3L}{3n}}{\binom{L}{n}^3}. \end{aligned}$$

We checked this formula numerically for systems with $2 \leq L \leq 12$, $1 \leq n \leq L/2$ and $x = 0, 0.1, \dots, 0.9$. For all these cases, we computed the sequence

$$a_k = \frac{1}{\epsilon} \frac{d^3}{d\gamma^3} \left(\frac{\langle 1 | (1 + \epsilon M(\gamma))^k | 1 \rangle}{\langle 1 | (1 + \epsilon M(\gamma))^{k-1} | 1 \rangle} - 1 \right)_{|\gamma=0} \quad (59)$$

for $1 \leq k \leq 200$ and $\epsilon = \frac{1}{(1+x)(n+1)}$, with γ being kept as a formal parameter. Using convergence acceleration techniques (fitting our sequences with a sum of two exponentials using Shanks transformation), we found that, for all the systems considered, the relative error on the third cumulant was at most 1.4×10^{-9} , which validates formula (58) for the third cumulant.

5.2. Some special cases

5.2.1. Totally asymmetric exclusion process. In the case of the totally asymmetric exclusion process ($x = 0$, the particles only hop to the right), using the binomial formulae (A.3),

(A.5) (twice), (A.8) and (A.9), all the sums in expression (58) for the third cumulant can be calculated, giving the result

$$\frac{E_3(x=0)}{p} = \frac{3L^2n(L-n)}{(L-1)(2L-1)} \frac{\binom{2L}{2n}^2}{\binom{L}{n}^4} - \frac{4L^2n(L-n)}{(L-1)(3L-1)} \frac{\binom{3L}{3n}}{\binom{L}{n}^3}, \quad (60)$$

which is the same as formula (13) in [10].

5.2.2. *Symmetric exclusion process.* In the case of the symmetric exclusion process ($x = 1$, the particles hop to the right and to the left with the same rate), we see that three of the terms of formula (58) are singular. But it turns out that all these singularities cancel, as well as all the constants terms, giving $E_3 = 0$. Indeed, we have the expansion near $x = 1$,

$$\frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} = \frac{4}{ij} \frac{1}{(x-1)^2} + \frac{4}{ij} \frac{1}{x-1} + \mathcal{O}(1). \quad (61)$$

The coefficients of $(x-1)^{-1}$ and of $(x-1)^{-2}$ are the same, so we only have one calculation to do to cancel both the divergent and the constant term of E_3 when $x \rightarrow 1$. Using the binomial formulae (A.4) and (A.11), we find that both terms vanish, leaving us with

$$E_3(x=1) = 0. \quad (62)$$

This result was expected: all the odd cumulants vanish for the symmetric exclusion process as the system is invariant if we exchange the right and the left. The probability density function of the current is even.

5.3. Scaling limit when $x \rightarrow 1$ as $L^{-1/2}$

The behavior of the system strongly depends on whether $x = 1$ or not. If $x = 1$, the rates at which the particles hop to the left and to the right are equal and the system belongs to the universality class of the Edwards–Wilkinson equation. If $x \neq 1$, the rates are not symmetric anymore and the system belongs to the universality class of the Kardar–Parisi–Zhang equation. The separation between these two regimes is given by the scaling $1-x \sim L^{-1/2}$ with fixed ratio $\rho = n/L$. This scaling is a rather natural separation between weak and strong asymmetry. Indeed, a particle makes $R \propto (1-x)\Delta t/L$ revolutions through the periodic lattice during a time interval Δt , from the value of the current. The natural time interval to be considered corresponds to the time necessary for the system to reach its stationary state. This time scales as L^z where the dynamical exponent z equals $3/2$ for the ASEP [7]. Thus, $1-x \sim L^{-1/2}$ is the scaling corresponding to a number of rotations $R \sim 1$ during the time necessary for the system to reach its stationary state. If $1-x \ll L^{-1/2}$, the number of rotations during this time is $\ll 1$ and the system behaves like the symmetric exclusion process. In contrast, if $1-x \gg L^{-1/2}$, the number of rotations is $\gg 1$ and the system behaves like the totally asymmetric exclusion process.

In [17], the scaling limit $1-x \sim 1/\sqrt{L}$ of the diffusion constant was calculated from the exact expression (54). It was found (up to a factor L^2 due to the fact that they calculated the current through a bond) that

$$\frac{\Delta}{p} \sim \Phi h_2(\Phi) \rho(1-\rho)L \quad \text{with} \quad h_2(\Phi) = 4 \int_0^\infty du \frac{u^2}{\tanh(\Phi u)} e^{-u^2} \quad (63)$$

and

$$x = e^{-f} \tag{64}$$

$$\Phi = \frac{f\sqrt{L\rho(1-\rho)}}{2}, \tag{65}$$

with Φ being held constant in this scaling limit. We will do the same here for the third cumulant in the same scaling limit. From definition (64) of f , we have

$$\frac{1+x^i}{1-x^i} = \frac{1}{\tanh\left(\frac{if}{2}\right)} = \frac{1}{\tanh\left(\frac{i\Phi}{\sqrt{L\rho(1-\rho)}}\right)}. \tag{66}$$

When L goes to infinity with fixed Φ , $f \rightarrow 0$, so that $f \sim 1-x$ and $(1-x)\sqrt{L}$ is also kept constant, which is precisely the wanted scaling limit. Letting L go to infinity with fixed ρ and Φ , expression (58) of the third cumulant becomes, using Stirling's approximation for the binomial coefficients,

$$\begin{aligned} \frac{E_3}{2\rho\Phi} \sqrt{\frac{\rho(1-\rho)}{L}} &\sim 6 \sum_{i>0} \sum_{j>0} \frac{(i^2+j^2) e^{-\frac{i^2+j^2}{L\rho(1-\rho)}}}{\tanh\left(\frac{i\Phi}{\sqrt{L\rho(1-\rho)}}\right) \tanh\left(\frac{j\Phi}{\sqrt{L\rho(1-\rho)}}\right)} \\ &- 6 \sum_{i>0} \sum_{j>0} \frac{(i^2+ij+j^2) e^{-\frac{i^2+ij+j^2}{L\rho(1-\rho)}}}{\tanh\left(\frac{i\Phi}{\sqrt{L\rho(1-\rho)}}\right) \tanh\left(\frac{j\Phi}{\sqrt{L\rho(1-\rho)}}\right)} - 3 \sum_{i>0} \frac{i^2 e^{-\frac{i^2}{L\rho(1-\rho)}}}{\tanh^2\left(\frac{i\Phi}{\sqrt{L\rho(1-\rho)}}\right)} \\ &+ \frac{3\sqrt{\pi}}{4} L^{3/2} (\rho(1-\rho))^{3/2} - \frac{2\pi}{3\sqrt{3}} \rho^2 (1-\rho)^2 L^2. \end{aligned} \tag{67}$$

We only wrote the dominant behavior of each term. The sums can now be written as Riemann integrals over the variables $u = i/\sqrt{L\rho(1-\rho)}$ and $v = j/\sqrt{L\rho(1-\rho)}$,

$$\begin{aligned} \frac{E_3}{p} &\sim 12\Phi(\rho(1-\rho))^{3/2} L^{5/2} \int_0^\infty \int_0^\infty du dv \frac{u^2+v^2-(u^2+uv+v^2) e^{-uv}}{\tanh(\Phi u) \tanh(\Phi v)} e^{-u^2-v^2} \\ &- 6\Phi\rho(1-\rho)L^2 \int_0^\infty du \frac{u^2}{\tanh^2 \Phi u} e^{-u^2} + \frac{3\sqrt{\pi}}{2} \Phi\rho(1-\rho)L^2 \\ &- \frac{4\pi}{3\sqrt{3}} \Phi(\rho(1-\rho))^{3/2} L^{5/2}. \end{aligned} \tag{68}$$

When $L \rightarrow \infty$, we get

$$\frac{E_3}{p} \sim -\Phi h_3(\Phi)(\rho(1-\rho))^{3/2} L^{5/2}, \tag{69}$$

with

$$h_3(\Phi) = \frac{4\pi}{3\sqrt{3}} - 12 \int_0^\infty \int_0^\infty du dv \frac{(u^2+v^2)-(u^2+uv+v^2) e^{-uv}}{\tanh(\Phi u) \tanh(\Phi v)} e^{-u^2-v^2}. \tag{70}$$

A numerical evaluation of $h_3(\Phi)$ tells us that it is a monotonic function (see figure 1). It grows from 0 for $\Phi = 0$ to $\frac{16\pi}{3\sqrt{3}} - 3\pi \approx 0.248\,8186\dots$ when $\Phi \rightarrow \infty$. More precisely, the behavior of $h_3(\Phi)$ in these limits is

$$h_3(\Phi) \sim \frac{2}{15} \Phi^2 \quad \text{when } \Phi \rightarrow 0 \tag{71}$$

$$h_3(\Phi) \rightarrow \frac{16\pi}{3\sqrt{3}} - 3\pi \quad \text{when } \Phi \rightarrow \infty, \tag{72}$$

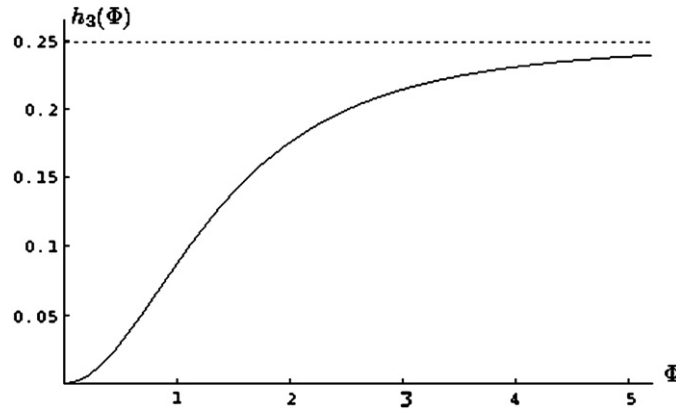


Figure 1. Plot of the function $h_3(\Phi)$ defined in (70), the asymmetry coefficient $\Phi \sim (1-x)\sqrt{L}$ being defined in (65). In the limit where the size of the system L goes to infinity and the rates asymmetry $1-x$ goes to 0 with fixed Φ , $h_3(\Phi)$ is proportional to the third cumulant of the current E_3 divided by Φ . The dashed line indicates the limit of $h_3(\Phi)$ when Φ goes to infinity, whose approximate value is 0.248 8186.

as can be seen using

$$\frac{1}{\tanh \Phi u \tanh \Phi v} \rightarrow 1 \quad \text{when } \Phi \rightarrow \infty \tag{73}$$

$$\frac{1}{\tanh \Phi u \tanh \Phi v} = \frac{1}{uv\Phi^2} + \frac{1}{3} \left(\frac{u}{v} + \frac{v}{u} \right) + \left(-\frac{u^3}{45v} + \frac{uv}{9} - \frac{v^3}{45u} \right) \Phi^2 + \mathcal{O}(\Phi^4) \tag{74}$$

when $\Phi \rightarrow 0$

and performing the integrals in polar coordinates.

5.3.1. Range of validity of the integral formula for the third cumulant. So far, we proved the integral formula (69) for the third cumulant in the scaling limit $1-x \sim 1/\sqrt{L}$. We will now see that it holds in fact for more general values of the asymmetry once we extract from it the factor $1-x$ which is in front of the finite-size expression (58). More precisely, we will show that

$$\frac{E_3(x)}{p} \sim -(1-x) \frac{h_3(\Phi)}{2} \rho^2 (1-\rho)^2 L^3 \quad \text{for } 1-x \sim 1/L^r \quad \text{with } 0 \leq r < 1, \tag{75}$$

with Φ being understood as the function of x, ρ and L defined in equations (64) and (65). The case $1-x \gg 1/\sqrt{L}$, which corresponds to taking the limit $\phi \rightarrow \infty$, is the easiest. In this limit, all the expressions remain bounded and nonzero, and $h_3(\Phi)$ simply converges to its $\Phi \rightarrow \infty$ limit (72) giving

$$\frac{E_3(1-x \gg 1/\sqrt{L})}{p} \sim -(1-x) \left(\frac{8\pi}{3\sqrt{3}} - \frac{3\pi}{2} \right) \rho^2 (1-\rho)^2 L^3 \quad \text{when } L \rightarrow \infty. \tag{76}$$

This is precisely equation (37) of [16] for the third cumulant when $x \neq 1$. If we take $x = 0$, we obtain the TASEP result [10].

The limit $1-x \ll 1/\sqrt{L}$, which corresponds to $\Phi \rightarrow 0$, is more difficult to obtain. From equation (71), we see that in this limit our integral expression for the third cumulant (69) is

of order $\Phi^3 L^{5/2}$. We also know that the third cumulant is equal to 0 when $x = 1$ ($\Phi = 0$), so that E_3 is at least proportional to Φ . However, it is not required to be of order Φ^3 . Thus, if Φ is small enough, subdominant terms in L proportional to Φ may become larger than the $\Phi^3 L^{5/2}$ term in equation (67). In appendix C, we show that the correction to equation (67) at order L^2 vanish. We also compute the correction of order $L^{3/2}$ (C.7) and show that it is equal, in the limit $\Phi \rightarrow 0$, to $2\Phi\rho^{3/2}(1-\rho)^{3/2}L^{3/2} = (1-x)\rho^2(1-\rho)^2L^2$. This is to be compared with the limit $\Phi \rightarrow 0$ of the leading order integral formula (69),

$$\frac{E_3(1-x \ll 1/\sqrt{L})}{p} \sim -\frac{(1-x)^3\rho^3(1-\rho)^3L^4}{60}. \quad (77)$$

If $1-x \sim 1/L$, these two expressions are of the same order. Thus, when $1-x \sim \sigma/L$ with fixed σ , the third cumulant is given by

$$\frac{E_3(1-x \sim \sigma/L)}{p} \sim \sigma\rho^2(1-\rho)^2L - \frac{\sigma^3}{60}\rho^3(1-\rho)^3L, \quad (78)$$

which agrees with [24]. No other subleading correction can become larger when $\Phi \rightarrow 0$ as each correction must be at least of order Φ .

We finally see that there are two interesting scaling limits for the third cumulant of the current: $1-x \sim 1/\sqrt{L}$ and $1-x \sim 1/L$. Both are in fact natural separations between weak and strong asymmetry. Indeed, if one takes the dynamical exponent to be equal to 2 (symmetric exclusion process) and not 3/2 in the discussion we made before equation (63), we find that the natural scaling becomes $1-x \sim 1/L$ and not $1-x \sim 1/\sqrt{L}$. In the scaling limit $1-x \sim 1/\sqrt{L}$, the third cumulant is given by equation (75) while in the scaling limit $1-x \sim 1/L$ it is given by equation (78). The first expression is valid as long as $1-x \sim 1/L^r$ with $0 \leq r < 1$. The second expression is valid for $1-x \sim 1/L^r$ with $r > 1/2$. In the region $1/2 < r < 1$, both expressions agree.

5.3.2. Normalized third cumulant: skewness. The skewness of the probability distribution of the current $\alpha_3 = E_3/\Delta^{3/2}$ is given, in the scaling limit $1-x \sim 1/\sqrt{L}$, by (see figure 2)

$$\frac{\alpha_3\sqrt{p}}{L} \sim -\frac{h_3(\Phi)}{\sqrt{\Phi}(h_2(\Phi))^{3/2}}. \quad (79)$$

In the limits $\Phi \rightarrow 0$ and $\Phi \rightarrow \infty$, it behaves as

$$\frac{\alpha_3\sqrt{p}}{L} \sim -\frac{\Phi^3}{15\sqrt{2}} \quad \text{when } \Phi \rightarrow 0 \quad (80)$$

$$\frac{\alpha_3\sqrt{p}}{L} \sim -\frac{\pi^{1/4}}{\sqrt{\Phi}} \left(\frac{16}{3\sqrt{3}} - 3 \right) \quad \text{when } \Phi \rightarrow \infty. \quad (81)$$

In particular, $\alpha_3\sqrt{p}/L$ goes to 0 when $\Phi \rightarrow 0$, which corresponds to the totally asymmetric case. Indeed for TASEP the diffusion constant Δ scales as $L^{3/2}$ while the third cumulant scales as L^3 , so that $\alpha_3 \sim L^{3/4} \ll L$.

We see that, at least in the scaling limit for which the asymmetry goes to 0 as $L^{-1/2}$, the skewness of the total current is negative. This indicates that the left tail of the probability distribution of the total current is larger than its right tail, which means that the current of the system is lower than its mean value with higher probability. This can be understood by a simple argument [10]: in order to reduce the current of the system, one has to reduce the speed of only one of the particles, as particles cannot overtake each other. In contrast, in order to increase the current of the system, one has to speed up all the particles as the current is limited by the slowest one. It is therefore natural to find a negative skewness for the current.

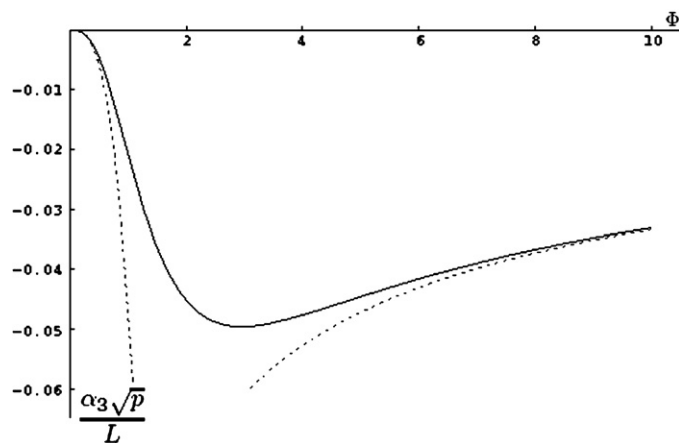


Figure 2. Plot of the skewness α_3 of the probability distribution of the current as a function of the asymmetry in the partially asymmetric exclusion process. The rates for hopping to the right and to the left are p and xp in the scaling limit for which the asymmetry coefficient $\Phi \sim (1-x)\sqrt{L}$ defined in (65) is held constant when the size of the system goes to infinity. We represented here $\alpha_3\sqrt{p}/L$, given in equation (79), as a function of Φ . The dashed curves are the asymptotics of the skewness when $\Phi \rightarrow 0$ ($\alpha_3 \sim \Phi^3$) and $\Phi \rightarrow \infty$ ($\alpha_3 \sim 1/\sqrt{\Phi}$).

6. Conclusion

We have presented here an explicit method allowing the calculation of the first cumulants of the total current for PASEP with a finite number of particles in a finite periodic one-dimensional lattice. This method, based on the underlying integrability of the model, relies on a perturbative resolution of the functional Bethe ansatz equation introduced in [18] to calculate the first two cumulants. We obtained exact formulae for the three first cumulants: in particular, we recovered the known formulae for the first two, but with significantly less calculations, and gave an explicit formula (58) for the third cumulant. This formula is much more complicated than the one for the diffusion constant, involving five terms, two of which are double sums. However, in the limit where the asymmetry x goes to 1 as the inverse of the square root of the size of the system, it takes a simple integral form (69) as does the diffusion constant in the same limit.

The perturbative solution formulated here can be used to get all the cumulants of the current for the weakly asymmetric exclusion process, for which the asymmetry scales as the inverse of the size of the system. It gives both the leading and sub-leading terms (work in preparation). It would be interesting to be able to do the same when the asymmetry scales as the inverse of the square root of the size of the system, where nice integral formulae exist for the diffusion constant and the third cumulant. We also think that these methods of functional Bethe ansatz could be used for other systems for which the Bethe ansatz equations look quite similar to those of PASEP. In particular for open exclusion process, for which the Bethe equations have been found recently [25, 26]. Or for exclusion process with different classes of particles, for which there has already been work on the fluctuations of the current [11].

As the asymmetric exclusion process belongs to the universality class of the Kardar–Parisi–Zhang equation, the cumulants of the current for this discrete model in some scaling limit should be characteristic of the KPZ equation. The ratio of the square of the third cumulant with the product of the second and the fourth cumulants was studied numerically by Monte Carlo simulations for TASEP and for two others one-dimensional growth models which also

belong to the KPZ universality class [27]. The numerical values obtained there were then compared to the large size limit of the exact result for TASEP derived by Bethe ansatz [10]. Something similar could be done for the third cumulant of PASEP we derived here.

For a random variable with Gaussian probability distribution, only the first and second cumulants are nonzero. The skewness of a probability distribution, defined as the third cumulant, divided by the second to the power 3/2, is then the first sign of non-Gaussianity visible on the cumulants. It indicates the asymmetry of the probability distribution. It was recently measured for the distribution of the current in mesoscopic one-dimensional systems subject to an electric driving field [28]. It would be interesting to find some real physical system for which the third cumulant of the current is given by our integral formula.

Acknowledgments

I thank O Golinelli and K Mallick for many useful discussions. I also thank K Mallick for his careful reading of the manuscript.

Appendix A. Binomial coefficients formulae

In the course of this paper, we used the following binomial coefficient formulae:

$$\sum_{k=0}^{n-1} \binom{L}{k} (-1)^k = -(-1)^n \frac{n}{L} \binom{L}{n} \quad (\text{A.1})$$

$$\sum_{k=0}^{n-1} k \binom{L}{k} (-1)^k = -(-1)^n n \frac{n-1}{L-1} \binom{L}{n} \quad (\text{A.2})$$

$$\sum_{k=1}^n \binom{L}{n+k} \binom{L}{n-k} = \frac{1}{2} \binom{2L}{2n} - \frac{1}{2} \binom{L}{n}^2 \quad (\text{A.3})$$

$$\sum_{k=1}^n k \binom{L}{n+k} \binom{L}{n-k} = \frac{n(L-n)}{2L} \binom{L}{n}^2 \quad (\text{A.4})$$

$$\sum_{k=1}^n k^2 \binom{L}{n+k} \binom{L}{n-k} = \frac{n(L-n)}{2(2L-1)} \binom{2L}{2n} \quad (\text{A.5})$$

$$\begin{aligned} \sum_{i>0} \sum_{j>0} & \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ & = \frac{1}{3} \binom{3L}{3n} - \binom{L}{n} \binom{2L}{2n} + \frac{2}{3} \binom{L}{n}^3 \end{aligned} \quad (\text{A.6})$$

$$\begin{aligned} \sum_{i>0} \sum_{j>0} (i+j) & \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ & = \frac{2}{3} \frac{n(L-n)}{L} \binom{L}{n} \binom{2L}{2n} - \frac{n(L-n)}{L} \binom{L}{n}^3 \end{aligned} \quad (\text{A.7})$$

$$\begin{aligned} \sum_{i>0} \sum_{j>0} ij \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ = -\frac{1}{3} \frac{n(L-n)}{3L-1} \binom{3L}{3n} + \frac{2}{3} \frac{n^2(L-n)^2}{L^2} \binom{L}{n} \end{aligned} \tag{A.8}$$

$$\begin{aligned} \sum_{i>0} \sum_{j>0} (i^2 + j^2) \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ = \frac{4}{3} \frac{n(L-n)}{3L-1} \binom{3L}{3n} - \frac{n(L-n)}{2L-1} \binom{L}{n} \binom{2L}{2n} - \frac{2}{3} \frac{n^2(L-n)^2}{L^2} \binom{L}{n} \end{aligned} \tag{A.9}$$

$$\begin{aligned} \sum_{i>0} \sum_{j>0} (i+j) \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} - \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ = -\frac{n(L-n)(L-2n)}{3L^2} \binom{L}{n} \end{aligned} \tag{A.10}$$

$$\begin{aligned} \sum_{i>0} \left(i + \frac{j}{2} \right) \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ = \left(\frac{n(L-n)}{L} - \frac{j}{2} \right) \binom{L}{n} \binom{L}{n+j} \binom{L}{n-j} \end{aligned} \tag{A.11}$$

$$\begin{aligned} \sum_{i>0} \left(i + \frac{j}{2} \right) \left[\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} - \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right] \\ = -\frac{j(L-2n)}{2L} \binom{L}{n} \binom{L}{n+j} \binom{L}{n-j}. \end{aligned} \tag{A.12}$$

We recall that the binomial coefficient $\binom{b}{a}$ is defined for an integer a as

$$\begin{aligned} \binom{b}{a} &= \frac{b(b-1)\cdots(b-a+1)}{a!} && \text{if } a > 0 \\ \binom{b}{0} &= 1 && \text{for } a = 0 \\ \binom{b}{a} &= 0 && \text{if } a < 0. \end{aligned} \tag{A.13}$$

In particular, if b is a positive integer,

$$\binom{b}{a} = 0 \quad \text{if } a > b. \tag{A.14}$$

Formulae (A.1) and (A.2) can be easily proved using Pascal's triangle formula $\binom{b-1}{a-1} + \binom{b-1}{a} = \binom{b}{a}$ (once for the first equation, twice for the second).

We will go on with formulae (A.4), (A.11) and (A.12), which are the easiest. Indeed, if we call F the function of the summation index that we want to sum (the summand), one can find a 'discrete primitive' G of F with respect to the summation variable, that is, if we call k

the summation variable, $F(k)$ can be written as

$$F(k) = G(k + 1) - G(k). \tag{A.15}$$

Knowing this expression for $F(k)$, it is then easy to do the summation for any range of summation. The F functions are respectively

$$F_{A.4}(k) = k \binom{L}{n+k} \binom{L}{n-k} \tag{A.16}$$

$$F_{A.11+A.12}(i) = \left(i + \frac{j}{2}\right) \binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} \tag{A.17}$$

$$F_{A.11-A.12}(i) = \left(i + \frac{j}{2}\right) \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \tag{A.18}$$

and the G functions are

$$G_{A.4}(k) = -\frac{(L-n+k)(n+k)}{2L} \binom{L}{n+k} \binom{L}{n-k} \tag{A.19}$$

$$G_{A.11+A.12}(i) = -\frac{(L-n+i+j)(n+i)}{2L} \binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} \tag{A.20}$$

$$G_{A.11-A.12}(i) = -\frac{(L-n+i)(n+i+j)}{2L} \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j}, \tag{A.21}$$

which can be checked easily, and gives us the wanted formulae. These discrete primitives can be found if F is hypergeometric in the summation variable using Gosper’s algorithm [29]. We used here an implementation of Gosper’s algorithm written by Peter Paule and Markus Schorn [30]. If a hypergeometric ‘discrete primitive’ exists, then Gosper’s algorithm will find it. For the other single variable summations, the algorithm fails: there is no hypergeometric discrete primitive.

We now move on to the last two simple sums (A.3) and (A.5). Here, the boundaries of the sums are important: with other boundaries, there might not be a simple formula for the sum. We will calculate them using a suitable generating function. Starting from the term in t^{2n} in the expansion in powers of t of $(1 + xt)^L(1 + yt)^L$, and keeping out of the sums the terms in t^n , we get

$$\left[\frac{(1 + xt)^L(1 + yt)^L}{x^n y^n} \right]_{(t^{2n})} = \binom{L}{n}^2 + \sum_{i>0} \left(\frac{x^i}{y^i} + \frac{y^i}{x^i} \right) \binom{L}{n+i} \binom{L}{n-i}. \tag{A.22}$$

In $x = y = 1$, it gives (A.3). If we take the derivative with respect to x and y and then set x and y to 1, we get (A.5). We can now sum (A.11) and (A.12) over j and use (A.3) and (A.4) to prove (A.7) and (A.10). Multiplying (A.11) by j and summing over j gives us a linear combination of (A.8) and (A.9), namely (A.8) + $\frac{1}{4}$ (A.9).

At this point, only (A.6) and either (A.8) or (A.9) are left. We will calculate them using a generating function, as for (A.3) and (A.5). Starting from the term in t^{3n} in the expansion in

powers of t of $(1 + xt)^L(1 + yt)^L(1 + zt)^L$, and keeping out of the sums the terms in t^n , we get

$$\begin{aligned} & \left[\frac{(1 + xt)^L(1 + yt)^L(1 + zt)^L}{x^n y^n z^n} \right]_{(t^{3n})} \\ &= \binom{L}{n}^3 + \binom{L}{n} \sum_{i>0} \left(\frac{x^i}{y^i} + \frac{x^i}{z^i} + \frac{y^i}{x^i} + \frac{y^i}{z^i} + \frac{z^i}{x^i} + \frac{z^i}{y^i} \right) \binom{L}{n+i} \binom{L}{n-i} \\ &+ \sum_{i>0} \sum_{j>0} \left[\left(\frac{x^i y^j}{z^{i+j}} + \frac{x^i z^j}{y^{i+j}} + \frac{y^i z^j}{x^{i+j}} \right) \binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} \right. \\ &\left. + \left(\frac{z^{i+j}}{x^i y^j} + \frac{y^{i+j}}{x^i z^j} + \frac{x^{i+j}}{y^i z^j} \right) \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j} \right]. \end{aligned} \tag{A.23}$$

In $x = y = z = 1$, it gives (A.6). If we take the derivative with respect to x and z and then set x, y and z to 1, we get another linear combination of (A.8) and (A.9), $-(A.8) - (A.9)$, proving the last two binomial formulae.

Appendix B. Calculation of the third cumulant

In this appendix, we calculate the third cumulant for systems for which $L \leq 2n$. It will be easier because all the sums will have their bounds between 0 and $2n$. In the end, using the particle-hole symmetry, we will note that our result holds in fact for all value of L .

B.1. Calculation of $g_3(t)$

Noting that for any function f , $[f(xt)]_{(a)} = x^a [f(t)]_{(a)}$, the recurrence equation (41) for $g(t)$ can be written in the more compact form

$$\begin{aligned} \tilde{g}_r(t) &= - \sum_{k=1}^{r-1} \sum_{\substack{b=1 \\ (b \neq n)}}^{(r-k)n} \frac{L^k}{k!} \frac{t^{-b}}{1 - x^{b-n}} [g_{r-k}]_{(-b)} + \sum_{k=1}^{r-1} \sum_{a=0}^L \sum_{b=1}^{(r-k)n} \sum_{c=1}^{kn} \\ &\times \frac{x^{c-n} \mathbb{1}_{a+n \neq b+c} \mathbb{1}_{a < b+c}}{(1 - x^{b+c-a-n}) t^{b+c-a}} (1 - \mathbb{1}_{a < b} - \mathbb{1}_{a < c}) [(1 - t)^L]_{(a)} [g_{r-k}]_{(-b)} [g_k]_{(-c)} \end{aligned} \tag{B.1}$$

with

$$\mathbb{1}_{u \neq v} = \begin{cases} 1 & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases} \quad \text{and} \quad \mathbb{1}_{u < v} = \begin{cases} 1 & \text{if } u < v \\ 0 & \text{if } u \geq v. \end{cases} \tag{B.2}$$

Introducing the notation

$$I[a, b] = [(1 - t)^L]_{(a)} [g_2]_{(-b)} [g_1]_{(-n)} = \nu(-1)^a \binom{L}{a} [g_2]_{(-b)}, \tag{B.3}$$

we get at order $r = 3$

$$\begin{aligned} \tilde{g}_3(t) &= - \sum_{\substack{j=1 \\ (j \neq n)}}^{2n} \frac{L t^{-j}}{1 - x^{j-n}} [g_2]_{(-j)} \\ &+ \sum_{i=0}^L \sum_{j=1}^{2n} \mathbb{1}_{i \neq j} \mathbb{1}_{i < j+n} (1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n}) t^{i-j-n} \frac{1 + x^{j-n}}{1 - x^{j-i}} I[i, j], \end{aligned} \tag{B.4}$$

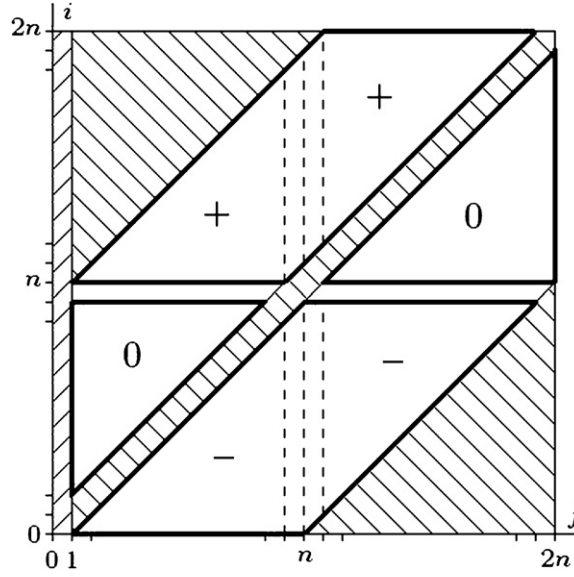


Figure B1. Range of summation over i and j given in equation (B.6). The + indicates that $\mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n}) = +1$, the - that $\mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n}) = -1$ and the 0 that $\mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n}) = 0$ due to a cancelation of a +1 with a -1. The hatched parts of the graph indicate terms that do not appear in the sums, because of $\mathbb{1}_{i \neq j}$, $\mathbb{1}_{i < j+n}$, or because of the sum boundaries. The boundaries of the +, - and 0 parts of the graph belong to these parts. The boundaries of the hatched parts do not belong to them.

which will give us the solution of equation (8) corresponding to the stationary state provided (40) is satisfied, that is

$$\sum_{i=0}^L \sum_{j=1}^{2n} \delta_{ij} (1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n})(1 + x^{j-n}) I[i, j] = L[g_2]_{(-n)} + \frac{L^2}{2} [g_1]_{(-n)}, \quad (\text{B.5})$$

which is easily proved. From now on, we will consider that $L \leq 2n$. Then, thanks to formula (48), we will only need the powers of t ranging from t^{-2n} to t^{-1} in $g(t)$ to be able to calculate the third order of the eigenvalue in the case $L \leq 2n$. We will note this $[\tilde{g}_3(t)]_{-2n}^{-1}$. In the double sum of equation (B.4), $\mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n})$ is either equal to -1, 0 or 1. The only indices (i, j) for which $\mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n})$ is nonzero are (see figure B1)

$$\begin{aligned} 0 \leq i \leq n-1 \quad i+1 \leq j \leq i+n \quad \mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n}) &= -1 \\ n \leq i \leq 2n \quad i-n+1 \leq j \leq i-1 \quad \mathbb{1}_{i < j+n}(1 - \mathbb{1}_{i < j} - \mathbb{1}_{i < n}) &= +1, \end{aligned} \quad (\text{B.6})$$

giving

$$\begin{aligned} [\tilde{g}_3(t)]_{-2n}^{-1} &= -L \sum_{\substack{j=1 \\ (j \neq n)}}^{2n} \frac{t^{-j} [g_2]_{(-j)}}{1 - x^{j-n}} + \sum_{i=n}^{2n} \sum_{j=i-n+1}^{i-1} \frac{1 + x^{j-n}}{1 - x^{j-i}} t^{i-j-n} I[i, j] \\ &\quad - \sum_{i=0}^{n-1} \sum_{j=i+1}^{i+n} \frac{1 + x^{j-n}}{1 - x^{j-i}} t^{i-j-n} I[i, j]. \end{aligned} \quad (\text{B.7})$$

We now split the sums over j according to the expression of $g_2(t)$,

$$\begin{aligned}
 [\tilde{g}_3(t)]_{-2n}^{-1} &= -L \sum_{j=1}^{n-1} \frac{t^{-j} [g_2]_{(-j)}}{1-x^{j-n}} - L \sum_{j=n+1}^{2n} \frac{t^{-j} [g_2]_{(-j)}}{1-x^{j-n}} + \sum_{i=n+1}^{2n-1} I[i, n] \frac{2t^{i-2n}}{1-x^{n-i}} \\
 &\quad - \sum_{i=0}^{n-1} I[i, n] \frac{2t^{i-2n}}{1-x^{n-i}} + \sum_{i=n}^{2n} \sum_{j=i-n+1}^{n-1} I[i, j] t^{i-j-n} \frac{1+x^{j-n}}{1-x^{j-i}} \\
 &\quad + \sum_{i=n}^{2n} \sum_{j=n+1}^{i-1} I[i, j] t^{i-j-n} \frac{1+x^{j-n}}{1-x^{j-i}} - \sum_{i=0}^{n-1} \sum_{j=i+1}^{n-1} I[i, j] t^{i-j-n} \frac{1+x^{j-n}}{1-x^{j-i}} \\
 &\quad - \sum_{i=0}^{n-1} \sum_{j=n+1}^{i+n} I[i, j] t^{i-j-n} \frac{1+x^{j-n}}{1-x^{j-i}}. \tag{B.8}
 \end{aligned}$$

We switch the sums over i and j in the double sums and then respectively make the following changes of variables in the sums:

Sum 1: $j \rightarrow n - j$	Sum 2: $j \rightarrow n + j$
Sum 3: $i \rightarrow n + i$	Sum 4: $i \rightarrow n - i$
Sum 5: $j \rightarrow n - j$ and $i \rightarrow n + i$	Sum 6: $j \rightarrow n + j$ then $i \rightarrow n + i + j$
Sum 7: $j \rightarrow n - j$ then $i \rightarrow n - i - j$	Sum 8: $j \rightarrow n + j$ and $i \rightarrow n - i$.

(B.9)

We obtain

$$\begin{aligned}
 [\tilde{g}_3(t)]_{-2n}^{-1} &= -L \sum_{j=1}^{n-1} \frac{[g_2]_{(-(n-j))}}{t^{n-j}} \frac{1}{1-x^{-j}} - L \sum_{j=1}^n \frac{[g_2]_{(-(n+j))}}{t^{n+j}} \frac{1}{1-x^j} \\
 &\quad + \sum_{i=1}^{n-1} \frac{I[n+i, n]}{t^{n-i}} \frac{2}{1-x^{-i}} - \sum_{i=1}^n \frac{I[n-i, n]}{t^{n+i}} \frac{2}{1-x^i} \\
 &\quad + \sum_{j=1}^n \sum_{i=0}^{n-1-j} \frac{I[n+i, n-j]}{t^{n-i-j}} \frac{1+x^{-j}}{1-x^{-i-j}} + \sum_{j=1}^n \sum_{i=1}^{n-j} \frac{I[n+i+j, n+j]}{t^{n-i}} \frac{1+x^j}{1-x^{-i}} \\
 &\quad - \sum_{j=1}^n \sum_{i=1}^{n-j} \frac{I[n-i-j, n-j]}{t^{n+i}} \frac{1+x^{-j}}{1-x^i} - \sum_{j=1}^n \sum_{i=1}^{n-j} \frac{I[n-i, n+j]}{t^{n+i+j}} \frac{1+x^j}{1-x^{i+j}}. \tag{B.10}
 \end{aligned}$$

Using expression (57) for $g_2(t)$ we can finally write, canceling a few terms and taking the lower boundaries of all the sums equal to 1,

$$\begin{aligned}
 [\tilde{g}_3(t)]_{-2n}^{-1} &= Lv^2 \sum_{j=1}^{n-1} \frac{\binom{L}{n+j} (-1)^{n-j}}{t^{n-j}} \frac{x^{2j}}{(1-x^j)^2} + Lv^2 \sum_{j=1}^n \frac{\binom{L}{n-j} (-1)^{n+j}}{t^{n+j}} \frac{x^j}{(1-x^j)^2} \\
 &\quad - \frac{2v^4}{L} \left(\sum_{j=1}^n \binom{L}{n+j} \binom{L}{n-j} \frac{1+x^j}{1-x^j} \right) \\
 &\quad \times \left(\sum_{i=1}^{n-1} \frac{\binom{L}{n+i} (-1)^{n+i}}{t^{n-i}} \frac{x^i}{1-x^i} + \sum_{i=1}^n \frac{\binom{L}{n-i} (-1)^{n-i}}{t^{n+i}} \frac{1}{1-x^i} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^n \sum_{i=1}^{n-1-j} \binom{L}{n+i} \binom{L}{n+j} \frac{v^3(-1)^{i+j}}{t^{n-i-j}} \frac{1+x^j}{1-x^j} \frac{x^{i+j}}{1-x^{i+j}} \\
 & + \sum_{j=1}^n \sum_{i=1}^n \binom{L}{n+i+j} \binom{L}{n-j} \frac{v^3(-1)^i}{t^{n-i}} \frac{1+x^j}{1-x^j} \frac{x^i}{1-x^i} \\
 & + \sum_{j=1}^n \sum_{i=1}^n \binom{L}{n-i-j} \binom{L}{n+j} \frac{v^3(-1)^i}{t^{n+i}} \frac{1+x^j}{1-x^j} \frac{1}{1-x^i} \\
 & + \sum_{j=1}^n \sum_{i=1}^{n-j} \binom{L}{n-i} \binom{L}{n-j} \frac{v^3(-1)^{i+j}}{t^{n+i+j}} \frac{1+x^j}{1-x^j} \frac{1}{1-x^{i+j}}.
 \end{aligned}$$

B.2. Calculation of E_3

Using (1) and (48), we express the third cumulant as

$$\frac{E_3}{6p} = (1-x) \sum_{l>0} \left[\binom{L}{l} (-1)^l \frac{l(n-l)}{L(L-1)} \tilde{g}_3(t) \right]_{(-l)} - (1-x) \frac{n^3}{3} \frac{L-n}{L-1} - n \frac{\Delta}{2p}. \tag{B.12}$$

We insert expression (B.11) of $\tilde{g}_3(t)$ into the latter equation, move the sum over l inside all the other sums containing t and finally set l to either $n-j, n+j, n-i, n+i, n-i-j$ or $n+i+j$ according to the sum using the $[\]_{-l}$. Rewriting

$$\frac{x^k}{1-x^k} = \frac{1}{2} \left(\frac{1+x^k}{1-x^k} - 1 \right) \tag{B.13}$$

and

$$\frac{1}{1-x^k} = \frac{1}{2} \left(\frac{1+x^k}{1-x^k} + 1 \right) \tag{B.14}$$

for all $k = i, j$ or $i+j$ in (B.11) and using

$$\frac{1+x^{i+j}}{1-x^{i+j}} \left(\frac{1+x^i}{1-x^i} + \frac{1+x^j}{1-x^j} \right) = 1 + \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} \tag{B.15}$$

and the binomial formulae (A.4) (two times), (A.8), (A.9) and (A.10), which allow us to calculate the sums not involving x , we get

$$\begin{aligned}
 \frac{(L-1)E_3}{6p(1-x)} & = L^2 \sum_{i>0} \sum_{j>0} \frac{\binom{L}{n+i} \binom{L}{n-i} \binom{L}{n+j} \binom{L}{n-j}}{\binom{L}{n}^4} (i^2 + j^2) \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} \\
 & - L^2 \sum_{i>0} \sum_{j>0} \frac{\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j}}{\binom{L}{n}^3} \\
 & \times \frac{i^2 + ij + j^2}{2} \frac{1+x^i}{1-x^i} \frac{1+x^j}{1-x^j} - L^2 \sum_{i>0} \frac{\binom{L}{n+i} \binom{L}{n-i}}{\binom{L}{n}^2} \frac{i^2}{2} \left(\frac{1+x^i}{1-x^i} \right)^2
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{L^2 n(L-n)}{4(2L-1)} \frac{\binom{2L}{n}}{\binom{L}{n}^2} - \frac{L^2 n(L-n)}{6(3L-1)} \frac{\binom{3L}{n}}{\binom{L}{n}^3} \\
 & + L^2 \sum_{j>0} \frac{1+x^j}{1-x^j} \left(\frac{\binom{L}{n+j} \binom{L}{n-j}}{\binom{L}{n}^2} \frac{j^2(L-2n)}{2L} \right. \\
 & + \left. \sum_{i>0} \frac{\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} - \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j}}{\binom{L}{n}^3} \left(\frac{j^2}{2} + ij \right) \right) \\
 & + nL^2 \sum_{j>0} \frac{1+x^j}{1-x^j} \left(\frac{\binom{L}{n+j} \binom{L}{n-j}}{\binom{L}{n}^2} \left(\frac{n(L-n)}{L} - \frac{j}{2} \right) \right. \\
 & - \left. \sum_{i>0} \frac{\binom{L}{n+i} \binom{L}{n+j} \binom{L}{n-i-j} + \binom{L}{n-i} \binom{L}{n-j} \binom{L}{n+i+j}}{\binom{L}{n}^3} \left(\frac{j}{2} + i \right) \right). \tag{B.16}
 \end{aligned}$$

The last two sums identically vanish thanks to the binomial formulae (A.11) and (A.12). This ends the proof of formula (58) for the third cumulant.

Appendix C. Subleading corrections to the third cumulant in the scaling limit

In this appendix, we compute the two first subleading corrections to the integral formula (69) for the third cumulant starting from the finite-size expression (58). We will need subleading corrections to the binomial coefficient formulae that we used to derive the integral expression (69). In particular, we will use

$$\frac{\binom{kL}{kn}}{\binom{L}{n}^k} = \sqrt{\frac{(2\pi\rho(1-\rho)L)^{k-1}}{k}} + \frac{(1-\rho+\rho^2)(k^2-1)}{12} \sqrt{\frac{(2\pi)^{k-1}(\rho(1-\rho)L)^{k-3}}{k^3}} + \mathcal{O}(L^{\frac{k-5}{2}}) \tag{C.1}$$

and

$$\begin{aligned}
 \frac{\binom{L}{n+i}}{\binom{L}{n}} & = \left(\frac{1-\rho}{\rho} \right)^i e^{-\frac{i^2+(1-2\rho)i}{2\rho(1-\rho)L}} \left(1 + \frac{1}{L^2} \left(\frac{(1-2\rho)i^3}{6\rho^2(1-\rho)^2} + \frac{(1-2\rho+2\rho^2)i^2}{4\rho^2(1-\rho)^2} + \dots \right) \right. \\
 & + \frac{1}{L^3} \left(-\frac{(1-3\rho+3\rho^2)i^4}{12\rho^3(1-\rho)^3} + \dots \right) + \frac{1}{L^4} \left(\frac{(1-2\rho)^2 i^6}{72\rho^4(1-\rho)^4} + \dots \right) + \mathcal{O}\left(\frac{1}{L^5}\right) \tag{C.2}
 \end{aligned}$$

The ... terms contain lower powers in i that come from subdominant corrections to the Stirling formula. They will not contribute to the corrections of E_3 we will calculate. In the following, we will also need an expansion of the difference between a Riemann sum and the corresponding limit integral. Let f be a function of two variables with a finite limit when any of the two variables goes to 0 and decaying exponentially at infinity. We want to calculate the difference between the sum

$$S = \epsilon^2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f(i, j) \tag{C.3}$$

and the integral

$$I = \int_0^\infty \int_0^\infty du dv f(u, v) \tag{C.4}$$

at order 2 in ϵ . Writing I as

$$\begin{aligned} I = \sum_{i=0}^\infty \sum_{j=0}^\infty \int_{i\epsilon}^{(i+1)\epsilon} \int_{j\epsilon}^{(j+1)\epsilon} du dv \left(f(i\epsilon, j\epsilon) \right. \\ \left. + (u - i\epsilon)\partial_u f(i\epsilon, j\epsilon) + (v - j\epsilon)\partial_v f(i\epsilon, j\epsilon) + \frac{(u - i\epsilon)^2}{2}\partial_u^2 f(i\epsilon, j\epsilon) \right. \\ \left. + (u - i\epsilon)(v - j\epsilon)\partial_u\partial_v f(i\epsilon, j\epsilon) + \frac{(v - j\epsilon)^2}{2}\partial_v^2 f(i\epsilon, j\epsilon) + \dots \right) \end{aligned} \tag{C.5}$$

and taking care of the fact that the sums begin with i and j equal to 1, we find

$$\begin{aligned} S = I - \frac{\epsilon}{2} \left(\int_0^\infty du f(u, 0) + \int_0^\infty dv f(0, v) \right) \\ - \epsilon^2 \left(-\frac{f(0, 0)}{4} + \frac{1}{12} \int_0^\infty du \partial_v f(u, 0) + \frac{1}{12} \int_0^\infty dv \partial_u f(0, v) \right) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{C.6}$$

Using the finite-size equation (58), we can now calculate subleading corrections to $E_3/(1-x)$. The term of order L^3 we get is the one we found previously (75). At the next order, we find that the $L^{5/2}$ term cancels out. Finally, the term of order L^2 is given by

$$\begin{aligned} \left[\frac{E_3}{1-x} \right]_{(L^2)} = 6\rho^2(1-\rho)^2 L^2 \int_0^\infty \int_0^\infty du dv \frac{(u^2+v^2)e^{-u^2-v^2} - (u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh(\Phi u)\tanh(\Phi v)} \\ + 3\rho(1-\rho)(1-2\rho+2\rho^2)L^2 \int_0^\infty \int_0^\infty du dv \\ \times \frac{(u^2+v^2)^2 e^{-u^2-v^2} - (u^2+uv+v^2)^2 e^{-u^2-uv-v^2}}{\tanh(\Phi u)\tanh(\Phi v)} \\ - \rho(1-\rho)(1-3\rho+3\rho^2)L^2 \int_0^\infty \int_0^\infty du dv \\ \times \frac{(u^2+v^2)(u^4+v^4)e^{-u^2-v^2} - (u^2+uv+v^2)^3 e^{-u^2-uv-v^2}}{\tanh(\Phi u)\tanh(\Phi v)} \\ + \frac{3}{4}\rho(1-\rho)(1-2\rho)^2 L^2 \int_0^\infty \int_0^\infty du dv \\ \times \frac{-u^2v^2(u+v)^2(u^2+uv+v^2)e^{-u^2-uv-v^2}}{\tanh(\Phi u)\tanh(\Phi v)} - \frac{20\pi}{27\sqrt{3}}\rho^2(1-\rho)^2 L^2 \\ - \frac{4\pi}{27\sqrt{3}}\rho(1-\rho)L^2 - \rho(1-\rho)L^2 \int_0^\infty \frac{(u^2 - \frac{u^4}{2})e^{-u^2}}{\Phi \tanh(\Phi u)}. \end{aligned} \tag{C.7}$$

This expression can be easily checked numerically: subtracting this expression plus the L^3 leading term to the finite-size expression (58) for various values of Φ , n and L , we find that the remaining term is of order $L \ll L^2$. Taking now the limit $\Phi \rightarrow 0$ in expression (C.7), we find that the divergent term vanishes as expected, leaving us with

$$\left[\frac{E_3}{1-x} \right]_{(L^2)} \rightarrow \rho^2(1-\rho)^2 L^2 \quad \text{when } \Phi \rightarrow 0. \quad (\text{C.8})$$

References

- [1] Liggett T M 1985 *Interacting Particle Systems* (New York: Springer)
- [2] Spohn H 1991 *Large Scale Dynamics of Interacting Particles* (New York: Springer)
- [3] Derrida B 1998 An exactly soluble non-equilibrium system: the asymmetric simple exclusion process *Phys. Rep.* **301** 65–83
- [4] Schütz G M 2001 Exactly solvable models for many-body systems far from equilibrium *Phase Transitions and Critical Phenomena* vol 19 (San Diego: Academic)
- [5] Golinelli O and Mallick K 2006 The asymmetric simple exclusion process: an integrable model for non-equilibrium statistical mechanics *J. Phys. A: Math. Gen.* **39** 12679–705
- [6] Dhar D 1987 An exactly solved model for interfacial growth *Phase Transit.* **9** 51
- [7] Gwa L-H and Spohn H 1992 Bethe solution for the dynamical-scaling exponent of the noisy Burgers equation *Phys. Rev. A* **46** 844–54
- [8] Derrida B, Evans M R, Hakim V and Pasquier V 1993 Exact solution of a one-dimensional asymmetric exclusion model using a matrix formulation *J. Phys. A: Math. Gen.* **26** 1493–517
- [9] Blythe R A and Evans M 2007 Nonequilibrium steady states of matrix-product form: a Solver’s guide *J. Phys. A: Math. Theor.* **40** R333–441
- [10] Derrida B and Lebowitz J L 1998 Exact large deviation function in the asymmetric exclusion process *Phys. Rev. Lett.* **80** 209–13
- [11] Derrida B and Evans M R 1999 Bethe Ansatz solution for a defect particle in the asymmetric exclusion process *J. Phys. A: Math. Gen.* **32** 4833–50
- [12] Bodineau T and Derrida B 2005 Distribution of current in non-equilibrium diffusive systems and phase transitions *Phys. Rev. E* **72** 066110
- [13] Boutillier C, François P, Mallick K and Mallick S 2002 A matrix Ansatz for the diffusion of an impurity in the asymmetric exclusion process *J. Phys. A: Math. Gen.* **35** 9703–30
- [14] Prähofer M and Spohn H 2002 Current fluctuations for the totally asymmetric simple exclusion process *In and Out of Equilibrium: Probability with a Physics Flavor (Progress in Probability* vol 51) (Boston: Birkhäuser) pp 185–204
- [15] Sasamoto T 2007 Fluctuations of the one-dimensional asymmetric exclusion process using random matrix techniques *J. Stat. Mech.* **P07007**
- [16] Lee D S and Kim D 1999 Large deviation function of the partially asymmetric exclusion process *Phys. Rev. E* **59** 6476–82
- [17] Derrida B and Mallick K 1997 Exact diffusion constant for the one dimensional partially asymmetric exclusion model *J. Phys. A: Math. Gen.* **30** 1031–46
- [18] Prolhac S and Mallick K 2008 Current fluctuations in the exclusion process and Bethe Ansatz *J. Phys. A: Math. Theor.* **41** 175002
- [19] Lebowitz J L and Spohn H 1999 A Gallavoti–Cohen type symmetry in the large deviation functional for stochastic dynamics *J. Stat. Phys.* **95** 333–65
- [20] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
- [21] Stroganov Y G 2001 The importance of being odd *J. Phys. A: Math. Gen.* **34** L179–85
- [22] Nepomechie R I 2003 Functional relations and Bethe Ansatz for the XXZ chain *J. Stat. Phys.* **111** 1363–76
- [23] Dorey P, Dunning C and Tateo R 2007 The ODE/IM correspondence *J. Phys. A: Math. Theor.* **40** R205–83
- [24] Appert C, Derrida B, Lecomte V and van Wijland F 2008 Universal cumulants of the current in diffusive systems on a ring *Preprint cond-mat/0804.2590*
- [25] de Gier J and Essler F H L 2005 Exact spectral gaps of the asymmetric exclusion process with open boundaries *Phys. Rev. Lett.* **95** 240601
- [26] Galleas W 2008 Functional relations from the Yang–Baxter algebra: eigenvalues of the XXZ model with non-diagonal twisted and open boundary conditions *Nucl. Phys. B* **790** 524–42

-
- [27] Derrida B and Appert C 1999 Universal large-deviation function of the Kardar–Parisi–Zhang equation in one dimension *J. Stat. Phys.* **94** 1–30
- [28] Reulet B 2005 Higher moments of noise *Proc. of Les Houches Summer School Session LXXXI* vol 81 (Amsterdam: Elsevier Science)
- [29] Petrovsek M, Wilf H S and Zeilberger D 1997 *A = B* (Wellesley, MA: A K Peters)
- [30] Paule P and Schorn M 1995 A Mathematica version of Zeilberger’s algorithm for proving binomial coefficient identities *J. Symb. Comput.* **20** 673–98